REDUCTION OF DISCRETE DYNAMICAL SYSTEMS
OVER GRAPHS

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In this paper we study phase space relations in a certain class of discrete dynamical systems over graphs. The systems we investigate are called Sequential Dynamical Systems (SDSs), which are a class of dynamical systems that provide a framework for analyzing computer simulations. Specifically, an SDS consists of (i) a finite undirected graph $Y$ with vertex set $\{1, 2, \ldots, n\}$ where each vertex has associated a binary state, (ii) a collection of $Y$-local functions $(F_i, Y)_{i \in \mathcal{V}(Y)}$ with $F_i : \mathbb{F}_2^\mathcal{V}(Y) \rightarrow \mathbb{F}_2^\mathcal{V}(Y)$ and (iii) a permutation $\pi$ of the vertices of $Y$. The SDS induced by (i)-(iii) is the map $[F_Y, \pi] = F_{\pi(n), Y} \circ \cdots \circ F_{\pi(1), Y}$.

The paper is motivated by a general reduction theorem for SDSs which guarantees the existence of a phase space embedding induced by a covering map between the base graphs of two SDSs. We use this theorem to obtain information about phase spaces of certain SDSs over binary hypercubes from the dynamics of SDSs over complete graphs. We also investigate covering maps over binary hypercubes, $Q_2^n$, and circular graphs, $\text{Circ}_n$. In particular we show that there exists a covering map $\phi : Q_2^n \rightarrow \text{Circ}_{n+1}$ if and only if $2^n \equiv 0 \pmod{n+1}$. Furthermore, we provide an interpretation of a class of invertible SDSs over circle graphs as right-shifts of length $n - 2$ over $\{0, 1\}^{2n-2}$. The paper concludes with a brief discussion of how we can extend a given covering map to a covering map over certain extended graphs.

Keywords: Sequential dynamical systems; graph morphisms; covering maps; phase space embeddings; reduction.

1. Introduction

This paper deals with a certain class of discrete dynamical systems called sequential dynamical systems [15], or SDSs for short. Essentially, an SDS consists of an undirected graph $Y$ with ordered vertex set $\{1, \ldots, n\}$, a multi-set of $Y$-indexed, local functions and a permutation $\pi \in S_n$. The SDS is induced by the composition of these local functions according to the permutation.

Before we proceed to define SDSs formally, let us first present the following example:

Example 1 (An SDS Over a Circle Graph). Let $Y = \text{Circ}_4$, the circle graph on four vertices, as shown in Fig. 1. To each vertex $i$ of the graph $Y$ we associate a
state \( x_i \in \mathbb{F}_2 \). Here \( \mathbb{F}_2 \) denotes the field with two elements, \{0, 1\} and where addition and multiplication are carried out modulo 2. The parity function\(^a\) \( \text{par}_3: \mathbb{F}_2^3 \to \mathbb{F}_2 \) is defined by \( \text{par}_3(x_1, x_2, x_3) = x_1 + x_2 + x_3 \). We define \( \text{Par}_i: \mathbb{F}_2^n \to \mathbb{F}_2^n, 1 \leq i \leq 4 \) by
\[
\begin{align*}
\text{Par}_1(x_1, x_2, x_3, x_4) &= (\text{par}_3(x_1, x_2, x_4), x_2, x_3, x_4), \\
\text{Par}_2(x_1, x_2, x_3, x_4) &= (x_1, \text{par}_3(x_1, x_2, x_3), x_3, x_4), \\
\text{Par}_3(x_1, x_2, x_3, x_4) &= (x_1, x_2, \text{par}_3(x_2, x_3, x_4), x_4), \\
\text{Par}_4(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, \text{par}_3(x_1, x_3, x_4)).
\end{align*}
\]

We apply these maps to the state \( x = (1, 1, 0, 0) \) in the order given by the identity permutation \( \pi = (1, 2, 3, 4) \), and at each stage we use the outcome of the previous functions as the input to the next function, i.e.
\[
(1, 1, 0, 0) \xrightarrow{\text{Par}_1} (0, 1, 0, 0) \xrightarrow{\text{Par}_2} (0, 1, 0, 0) \xrightarrow{\text{Par}_3} (0, 1, 1, 0) \xrightarrow{\text{Par}_4} (0, 1, 1, 1).
\]

Thus we have \([\text{Par}_{\text{Circ}_4}, \pi](1, 1, 0, 0) = \text{Par}_4 \circ \text{Par}_3 \circ \text{Par}_2 \circ \text{Par}_1(1, 1, 0, 0) = (0, 1, 1, 1)\). By iterating the map \( \phi = [\text{Par}_{\text{Circ}_4}, \pi] \) we obtain the orbit of \((1, 1, 0, 0)\) which in this case is the directed graph
\[
\begin{align*}
(1, 1, 0, 0) \xrightarrow{\phi} & (0, 1, 1, 1) \\
\phi \quad & \phi \\
(0, 0, 0, 1) \xleftarrow{\phi} & (0, 1, 1, 1).
\end{align*}
\]

We can now formally define SDSs. An SDS consists of the following data: (a) a finite, labeled graph \( Y \) with vertex set \{1, \ldots, n\} where each vertex \( i \) has a state \( x_i \in \mathbb{F}_2 \), (b) a vertex-labeled sequence of functions \( (F_{i,Y}: \{0, 1\}^n \to \{0, 1\}^n)_i \), and (c) a permutation \( \pi \in S_n \). Here \( S_n \) denotes the symmetric group over \( n \) letters, i.e. the group formed by all bijections of the set \{1, \ldots, n\}.\(^b\) The function \( F_{i,Y} \)

\(^a\)The parity function is the rule numbered 150 in the usual enumeration scheme of elementary CA rules.

\(^b\)In this paper we avoid cycle notation for permutations. Thus, for a graph on vertices \{1, 2, 3, 4\} the permutation \( \pi = (1, 2, 3, 4) \) denotes the identity permutation.
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updates the state of vertex $i$ based on the states of vertex $i$ and its $Y$-neighbors and leaves the states of all other vertices fixed. In this sense, each $F_{i,Y}$ is $Y$-local. The permutation $\pi$ represents a reordering of the vertices of $Y$ according to which the functions $F_{i,Y}$ are applied. By composing the functions $F_{i,Y}$ in the order given by $\pi$ we obtain the sequential\footnote{The term sequential is motivated by the successive application of the local functions.} dynamical system (SDS)

$$[F_Y, \pi] = F_{\pi(n)} \circ \cdots \circ F_{\pi(1)} : \{0, 1\}^n \to \{0, 1\}^n.$$  

In this paper we will study certain phase space relations between SDSs. The phase space of an SDS $[F_Y, \pi]$ is the directed graph $\Gamma[F_Y, \pi]$ with vertex set $\mathbb{F}_2^n$ and edge set $\{(x, [F_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$. We will call the cycles in this directed graph orbits. In order to formalize what we mean by phase space relations between two SDSs we will introduce the concept of an SDS morphism in the next section. Intuitively, a morphism between two SDSs should relate their basic constituents, i.e. their base graphs, their phase spaces and their underlying schedules.

Originally, SDSs were introduced to provide a framework for a theory of (computer) simulation, see Refs. 1, 4 and 15. In fact, SDSs capture the generic structure of computer simulations in a straightforward way: simulations typically have entities (the labeled vertices $\{1, \ldots, n\}$) that interact in some way and update their states (state transition functions $F_{i,Y}$). The interaction is typically localized (1-neighborhood in the graph $Y$), and individual state updates are executed according to some scheduling mechanism (functions $(F_{i,Y})_i$ are executed in the order given by the permutation $\pi$).

In the language of agent-based simulations (see Ref. 8) we may interpret the pair $(x_i, F_{i,Y})$ as an agent, and the 1-neighborhood at vertex $i$ in $Y$ as this agent’s communication links to other agents. The permutation serves as the update schedule of the system and specifies the order in which the agents act.

One of the first topics of SDS research was how a change in update schedule affects the SDS dynamics when the base graph $Y$ and local functions $(F_{i,Y})_i$ are kept fixed [4]. It turned out that the equivalence classes of update schedules, i.e. sets of update schedules for which the corresponding SDSs are the same, directly correspond to the acyclic orientations\footnote{An acyclic orientation of an undirected graph is an orientation of its edges such that the resulting directed graph is cycle-free.} of the base graph itself [16]. Moreover, the automorphism group of the graph acts naturally on the associated SDS phase space digraph [15, 19]. The above studies led to a combinatorial upper bound for the number of non-equivalent SDSs (in the sense of having non-isomorphic phase spaces) that can be obtained through rescheduling [15, 19]. Among other SDS research
topics that have been investigated we mention:

• Phase space structure: Periodic orbits, fixed points, invertibility [1–3,15,19].
• Scheduling: Equality and equivalence of SDS under schedule changes; scheduling
generalizations [2,9,15,17].
• SDS morphisms: Generalized equivalence of SDS [11–13,18,19].

Many SDS results remain valid even when the states belong to an arbitrary
finite field. Some results (e.g. on equivalence [2]) hold even when states are taken
from \( \mathbb{R} \) or \( \mathbb{C} \), and this is indeed the case for the results presented herein. Work has
also been done to extend schedules from permutations to words over the vertices of
the base graph, see Ref. 9.

This paper is motivated by the work [18, 19] which investigates how covering
maps between the corresponding base graphs naturally induce SDS morphisms.
(A covering map of graphs is a graph morphism which is locally bijective.) In
particular, Ref. 19 contains a general result on SDS phase space embeddings. While
Ref. 19 is primarily concerned with proving the existence of these SDS morphisms,
this paper provides examples, applications and extensions of this theory. The result
in Ref. 19 allows us for example to prove the existence of certain phase space
features, such as properties of periodic orbits, of a given SDS by means of smaller
or reduced SDSs. Furthermore, this paper extends the results of Ref. 18 on binary
hypercubes to circle graphs, \( \text{Circ}_n \). The results on circle graphs are closely related
to Ref. 18 where the author presents conditions under which these SDS reductions
can be obtained for SDSs over hypercubes. Finally, we will present certain general
extension principles for reductions. Further extension results can also be found in
the upcoming paper [14].

Typically, the phase space of an SDS has more than one attractor and conse-
quently, an orbit only constitutes parts of this phase space. (An attractor of an SDS
\([F_Y,\pi]\) is the unique orbit (which may only contain one vertex) of a component
in its phase space.) In the context of computer simulations this means that there will
be valid states or regimes that are never realized. Accordingly, we are interested in
constructing a smaller simulation system that has identical or specifically related
dynamics as the original system in these regimes, and that has as few other states
as possible.

In this paper we address this construction by establishing an embedding of
SDS phase spaces when certain conditions are satisfied. Explicitly, we will show
how to relate an SDS \( F \) over a graph \( Y \) and an SDS \( G \) over a smaller graph \( Z \) if
there exists a covering map \( \phi: Y \to Z \). This idea can be illustrated as follows: Let
\( Y = Q_3^2 \) (the binary, 3-dimensional hypercube) and \( Z = K_4 \) (complete graph over
4 vertices) (see Fig. 2). We will consider sequential dynamical systems over \( Q_2^3 \) and
\( K_4 \) induced by the parity function, \( \text{par}_4(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4 \) (modulo
2). We first observe that there exists a covering map \( p: Q_2^3 \to K_4 \) that is obtained by
identifying spatial diagonals in the cube, that is, \( p^{-1}(\{i\}) = \{i-1, 8-i\}, 1 \leq i \leq 4 \).
The map \( p \) naturally induces an embedding \( \tau: F_2^3 \to F_2^8 \) given by \( (\tau(x))_k = x_{p(k)} \)
with coordinate indices $1 \leq k \leq 8$, i.e.

$$\tau(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, x_3, x_2, x_1).$$

Let $\pi = (1, 2, 3, 4) \in S_4$, and let $\pi_p = (0, 7, 1, 6, 2, 5, 3, 4) \in S_8$. We obtain the commutative diagram

$$\begin{array}{ccc}
F_2^4 & \xrightarrow{\text{Par}_{K_4, (1,2,3,4)}} & F_2^4 \\
\downarrow{\tau} & & \downarrow{\tau} \\
F_2^3 & \xrightarrow{\text{Par}_{Q_2^3, \pi_p}} & F_2^8 \\
\end{array}$$

Using the notation in the previous example we have $[\text{Par}_{K_4, (1, 2, 3, 4)}](1, 0, 0, 0) = (1, 1, 0, 0)$ and obtain the following orbit

$$(1, 0, 0, 0) \rightarrow (1, 1, 0, 0) \rightarrow (0, 1, 1, 0)$$

$$(0, 0, 0, 1) \leftarrow (0, 0, 1, 1)$$

Note that we have the commutative diagram

$$\begin{array}{ccc}
(1, 0, 0, 0) & \xrightarrow{\text{Par}_{K_4, (1,2,3,4)}} & (1, 1, 0, 0) \\
\downarrow{\tau} & & \downarrow{\tau} \\
(1, 0, 0, 0, 0, 0, 0, 0, 1) & \xrightarrow{\text{Par}_{Q_2^3, \pi_p}} & (1, 1, 0, 0, 0, 0, 0, 1, 1) \\
\end{array}$$

By applying the map $\tau$ to the orbit of $(1, 0, 0, 0)$ under $[\text{Par}_{K_4, (1, 2, 3, 4)}]$ we obtain the orbit of $(1, 0, 0, 0, 0, 0, 0, 1)$ under $[\text{Par}_{Q_2^3, \pi_p}]$:

$$(1, 0, 0, 0, 0, 0, 0, 1) \rightarrow (1, 1, 0, 0, 0, 0, 1, 1) \rightarrow (0, 1, 1, 0, 0, 1, 1, 0)$$

$$(0, 0, 0, 1, 1, 0, 0, 0) \leftarrow (0, 0, 1, 1, 1, 0, 0)$$

Fig. 2. The graph $K_4$ is a covering image of the graph $Q_2^3$. 

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Further calculations show that the entire phase space of the SDS $[\text{Par}_{K_4}, (1, 2, 3, 4)]$ is embedded in the phase space of $[\text{Par}_{Q^2_p}, \pi_p]$.

1.1. Organization of the paper and main results

In Sec. 2 we present the Reduction Theorem and as our first main result we give in Proposition 1 an application which allows us to obtain information about the phase space of certain SDSs over $n$-cubes (hypercubes) from the dynamics of SDSs over complete graphs. Section 3 deals with results on the existence of covering maps from circle graphs and hypercubes. In particular, these results allow us to derive reduced dynamical systems as shown in the above example. We also present a number-theoretic result on the existence of a covering map from the $n$-dimensional hypercube to a complete graph. The significance of this is that SDSs over complete graphs are usually much more amenable to analysis. As our second main result we provide via Proposition 3 insight into the structure of invertible SDSs over circular graphs. We conclude with a number of examples of covering maps from non-regular graphs, and we provide an outline of an extension principle that allows us to extend a covering map from a given graph $Y$ to a covering map over certain extended graphs $Y'$.

2. The Reduction Theorem

Some graph classes have already been introduced, but we repeat them here for completeness: We will write $K_n$ for the complete graph over $n$ vertices, $\text{Circ}_n$ for the circle graph over $n$ vertices, and $Q^n_\alpha$ for the generalized $n$-cube (or hypercube) over the alphabet $\{0, 1, \ldots, \alpha - 1\}$.

Let $Y$ be a labeled graph with vertex set $v[Y] = N_n = \{1, 2, 3, \ldots, n\}$, which we write as $Y < K_n$. The edge-set of $Y$ is denoted by $e[Y]$. We use $|Y|$ as a shorthand for $|v[Y]|$. A morphism between graphs $Y$ and $Y'$ is a pair $\phi = (\phi_1, \phi_2)$ with $\phi_1: v[Y] \rightarrow v[Y']$ and $\phi_2: e[Y] \rightarrow e[Y']$ such that

$$\forall e = \{i, j\} \in e[Y]: \phi_2(e) = \{\phi_1(i), \phi_1(j)\}.$$ 

Thus, adjacent vertices in $Y$ are mapped to (i) adjacent vertices in $Y'$ or (ii) to the same vertex in $Y'$. The image graph $Y'$ may, in general, contain loops. A morphism of directed graphs also preserves the direction of edges. A graph morphism $\phi: Y \rightarrow Y'$ is locally bijective (respectively surjective) if the map

$$\forall i \in v[Y]: \phi|_{B_1,Y}(i) = \{\phi_1(i), \phi_1(j)\}$$

is bijective (respectively surjective), and where $B_1,Y(i) = \{i\} \cup \{j | \{i, j\} \in e[Y]\}$ is the 1-neighborhood of $i$ in $Y$. In the following we will use the term covering map.

The former SDS has one fixed point and three orbits of length 5 while the latter SDS has one fixed point and 51 orbits of length 5.

In fact, the example above is a particular instance of Reduction Theorem in Sec. 2.
instead of locally bijective graph morphism. Note that a covering map is generally not bijective (see Fig. 2). We define $\tilde{B}_{1,Y}(i)$ as the sequence of elements from the set $B_{1,Y}(i)$ in increasing (natural) order, but with vertex $i$ moved to the front. We write this as

$$\tilde{B}_{1,Y}(i) = (i, j_1, \ldots, j_{\delta_i}). \quad (2.1)$$

A function $f: E^k \rightarrow F$, where $E$ and $F$ are vector spaces, is quasi-symmetric if for all $x \in E^k$ and all permutations $\sigma \in \mathbb{S}_k$ we have $f(\sigma \cdot x) = f(x)$ where $\sigma \cdot x$ is the permutation action on $k$-tuples given by $\sigma \cdot x = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. We write $\text{QSymm}(E^n, F)$ for the set of all quasi-symmetric functions from $E^n$ to $F$. In the context of SDS these functions are the ones that do not depend on the order of neighbor states.

Let $Y < K_n$ be given. To each vertex $i$ of $Y$ we associate a state $x_i \in F_2$, and we write $x = (x_1, x_2, \ldots, x_n)$ for the system state. Let $\delta_i$ denote the degree of the vertex $i$ and set $d = \max_i \delta_i$. For each $k = 1, \ldots, d + 1$ we have a function $f_k \in \text{QSymm}(F_2^k, F_2)$. From these functions we form the sequence $(F_{i,Y}: F_2^k \rightarrow F_2^l)_{i=1}^n$ by

$$F_{i,Y}(x) = (x_1, \ldots, x_{i-1}, f_{\delta_i+1}(x_i, x_{j_1}, \ldots, x_{j_{\delta_i}}), x_{i+1}, \ldots, x_n). \quad (2.2)$$

The function $F_{i,Y}$ updates the state of vertex $i$ and leaves all other states fixed. We set $F_Y = (F_{i,Y})_i$, and define the map $[F_Y, \pi]: S_n \rightarrow \text{Map}(F_2^n, F_2^n)$ by

$$[F_Y, \pi] = \prod_{i=1}^n F_{\pi(i),Y} = F_{\pi(n),Y} \circ \cdots \circ F_{\pi(1),Y}. \quad (2.3)$$

We can now restate the definition of SDSs:

**Definition 1 (Sequential Dynamical System).** Let $Y < K_n$, let $(f_k)_k, 1 \leq k \leq d(Y) + 1$ where $f_k \in \text{QSymm}(F_2^k, F_2)$, and let $\pi \in S_n$. The sequential dynamical system (SDS) over $Y$ induced by $(f_k)_k$ with respect to the ordering $\pi$ is $[F_Y, \pi]$.

The phase space of an SDS $[F_Y, \pi]$ is the digraph $\Gamma[F_Y, \pi]$ with vertex set $F_2^n$ and directed edges $\{(x, [F_Y, \pi](x)) \mid x \in F_2^n\}$. Finally, let $H$ be a group acting on a graph $Y$.\(^b\) The orbit graph $H \setminus Y$ is the graph given by

$$\begin{align*}
 v[H \setminus Y] &= \{H(i) \mid i \in v[Y]\} \\
 e[H \setminus Y] &= \{H(e) \mid e \in e[Y]\}.
\end{align*}$$

2.1. Reduction of SDS

Our interpretation of reduction of an SDS consists in relating its phase space to the phase space of another SDS, typically an SDS defined over a smaller or simpler

\(^a\)We also refer to the ordering in an SDS as the schedule or update schedule.

\(^b\)The group $H$ acts on the graph $Y$ if there is a homomorphism $\phi: H \rightarrow \text{Aut}(Y)$ where $\text{Aut}(Y)$ is the automorphism group of $Y$. 
In order to make this precise we need to define the concept of morphism for SDS. However, before doing so, we look at some classical equivalence concepts and results. The notion of equivalence is central to dynamical system theory. For example, two maps $F: N \to N$ and $G: M \to M$ over topological spaces $M$ and $N$ are called topologically conjugate [6,7,10] if there exists a homeomorphism $h: N \to M$ such that $h \circ F = G \circ h$, that is, the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{F} & N \\
\downarrow{h} & & \downarrow{h} \\
M & \xrightarrow{G} & M
\end{array}
\]

commutes. It is easily seen that the map $h$ (the conjugacy between $F$ and $G$) maps orbits of $F$ to orbits of $G$. The concept of topological conjugation is also what lies at the heart of the celebrated Hartman/Grobman linearization theorem. In the case of a differentiable map $F: \mathbb{R} \to \mathbb{R}$ it states that if $F'(p) = \lambda$ with $|\lambda| \neq 0,1$ then there exists a local homeomorphism $h$ which conjugates $F$ and the linear map $G(x) = \lambda x$ on suitable open subsets containing $p$ and 0, respectively. In practice, this means that whatever we can deduce about the linear system $G$ around the state 0 also applies to the original system $F$ around the state $p$.

Based on this we may define a morphism from an SDS $[G_Z,\sigma]$ to an SDS $[F_Y,\pi]$ to be an injective digraph morphism $h: \Gamma[G_Z,\sigma] \to \Gamma[F_Y,\pi]$. However, an SDS has more structure than an arbitrary map $F: \mathbb{F}_2^Z \to \mathbb{F}_2^Y$. It is desirable to be able to relate this structure, so we define an SDS morphism as follows:

**Definition 2 (SDS Morphism).** Let $[F_Y,\pi]$ and $[G_Z,\sigma]$ each be an SDS. An SDS morphism between $[F_Y,\pi]$ and $[G_Z,\sigma]$ is a triple $(\phi, \Phi, \eta)$ where $\phi: Y \to Z$ is a graph morphism, $\Phi: \Gamma[G_Z,\sigma] \to \Gamma[F_Y,\pi]$ is digraph morphism and $\eta: S|_Z \to S|_Y$ is a map taking $Z$-schedules into $Y$-schedules.

As it stands it will typically be hard to verify if a map of this kind is an SDS morphism. However, and as we will show, if we have a covering map $\phi: Y \to Z$, we can derive a schedule map $\eta_\phi: S|_Z \to S|_Y$ and a state map $\tau_\phi: \mathbb{F}_2^Z \to \mathbb{F}_2^Y$ such that $\tau_\phi$ naturally induces a digraph morphism $\Phi_\phi$. In other words, given a covering map, we can explicitly construct SDS morphisms. We will show the constructions of all these maps in the following.

### 2.2. Construction of SDS morphisms from covering maps

Let $\phi: Y \to Z$ be a graph morphism. Unless otherwise stated, we will assume that the local functions of the $Y$-SDS and the $Z$-SDS satisfy the following condition:

\[ \forall i \in v[Z]: \forall j \in \phi^{-1}(i): F_{Z,i} = F_{Y,j}. \]  

(2.4)
We first relate update schedules for \( Y \) and \( Z \) via \( \phi \). Assume \( |Y| = n \) and \( |Z| = m \), and let \( \phi^{-1}(i) = \{i_1, \ldots, i_l\} \) with \( i_1 < \cdots < i_l \) and \( 1 \leq i \leq m \). Define the map \( \eta_\phi : S_m \rightarrow S_n \) by
\[
\eta_\phi(\pi = (\pi_1, \pi_2, \ldots, \pi_m)) = (\pi_{i_1}, \ldots, \pi_{i_{l+1}}, \ldots, \pi_{m+1}, \ldots, \pi_{1l+m}).
\] (2.5)
As an illustration, we consider the covering map \( \phi : Q_2^3 \rightarrow K_4 \) (see Fig. 2) and obtain
\[
\eta_\phi(4, 3, 2, 1) = (3, 4, 2, 5, 1, 6, 0, 7).
\]
In a similar way, we define the map \( \tau_\phi : F^m_2 \rightarrow F^n_4 \) by
\[
(\tau_\phi(x))_k = x_{\phi(k)}.
\] (2.6)
Given a covering map \( \phi : Y \rightarrow Z \) we can obtain an SDS morphism between SDS over \( Y \) and SDS over \( Z \) as follows:

**Theorem 1** ([19]). Let \( Y \) and \( Z \) be loop-free connected graphs, let \( \phi : Y \rightarrow Z \) be a covering map, and let \( (f_i) \) be a fixed sequence of quasi–symmetric Boolean functions (as in Eqn. (2.2)). Then the map \( \phi \) induces an SDS morphism. That is, we have a commutative diagram

\[
\begin{array}{c}
F^m_2 \xrightarrow{[F_2, \pi]} F^m_2 \\
\downarrow \tau_\phi \downarrow \\
F^n_4 \xrightarrow{[F_4, \eta_\phi(\pi)]} F^n_4 \\
\end{array}
\]

**Example 2.** For the covering map \( \phi : Q_2^3 \rightarrow K_4 \) we take \( \pi = (1, 2, 3, 4) \) and set \( \sigma = \eta_\phi(\pi) = (0, 7, 1, 6, 2, 5, 3, 4) \). Theorem 1 now gives us an embedding of the phase space of \( \text{Min}_{K_4} \) into the phase space of \( \text{Min}_{Q_2^3} \). Here \( \text{Min}_Y \) denotes the sequence of local functions induced by the minority function \( \text{min}_4 : F^2_2 \rightarrow F^2_2 \) on the graph \( Y \). The map \( \text{min}_4 \) applied to a state \( x \) gives 0 if \( x \) contains less than two zeros and gives 1 otherwise. In Ref. 1 we have shown that \( \text{Min}_{K_4} \) has exactly two 5-orbits and no fixed points. The two 5-orbits are shown in the left column of Fig. 3 where we encoded the states in decimal, i.e.
\[
\xi_i : F^i_2 \rightarrow \mathbb{N}, \quad \xi_i(x_1, \ldots, x_i) = \sum_{j=0}^{i} x_j \cdot 2^{i-1},
\]
so that, e.g. \((1, 1, 0, 1) \mapsto 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 4 + 1 \cdot 8 = 11 \). In view of Fig. 3, it is obvious that \( \Gamma[\text{Min}_{K_4}, \pi] \) is embedded in \( \Gamma[\text{Min}_{Q_2^3}, \sigma] \).

We remark that \( \text{Min}_{Q_2^3} \) has two fixed points in addition to the two 5-orbits shown in the right column in Fig. 3. These fixed points are related by the \( Q_2^3 \) graph automorphism \( \gamma = (07)(16)(25)(34) \) and consequently, so are their transients. Stated differently, the two components in \( \Gamma[\text{Min}_{Q_2^3}, \sigma] \) containing the fixed points are isomorphic. The structure of these components is shown in detail in Fig. 4.
Fig. 3. The left column shows the phase space of $[\text{Min}_{K_4}, \text{id}]$. The middle column shows the image of $\Gamma[\text{Min}_{K_4}, \text{id}]$ under the embedding map. Finally, in the right column we have shown the components of $\Gamma[\text{Min}_{Q_3}, \sigma]$ that embed $\Gamma[\text{Min}_{K_4}, \text{id}]$. Note that we have encoded binary tuples in decimal (e.g. $(0,1,0,0)$ is encoded as 2) to avoid clutter.

Fig. 4. The structure of the components in $\Gamma[\text{Min}_{Q_3}, \sigma]$ containing a fixed point. A single filled dot depicts a single state. A circled number $i$ depicts a state with $i$ direct predecessors, none of which have any predecessors themselves.
Given a covering map other than an isomorphism \( \phi: Y \to Z \), it will usually be more feasible to analyze the SDS over \( Z \) than the corresponding SDS over \( Y \). The following proposition can be viewed as a paradigm on how to obtain information about certain phase space features of complicated SDS.

**Proposition 1.** Let \( n \geq 3 \) and assume \( 2^n \equiv 0 \mod n + 1 \). Then there exists a covering map \( \phi: Q_2^n \to K_{n+1} \), and any SDS of the form \([\text{Par}_{Q_2^n}, \eta_\phi(\pi)]\) with \( \pi \in S_{n+1} \) has a periodic orbit of length \( n + 2 \).

**Proof.** In Sec. 3 we will show that the condition \( 2^n \equiv 0 \mod n + 1 \) is both sufficient and necessary for the existence of a covering map \( \phi: Q_2^n \to K_{n+1} \). Given this fact, we may again apply Theorem 1 and we conclude that the phase space of \([\text{Par}_{K_{n+1}}, \pi]\) is embedded in the phase space of \([\text{Par}_{Q_2^n}, \eta_\phi(\pi)]\). The proposition will follow if we can prove the existence of a periodic orbit of length \( n + 2 \) for \([\text{Par}_{K_{n+1}}, \pi]\).

Without loss of generality we may choose \( \pi = \text{id}_{n+1} = (1, 2, \ldots, n+1) \). This is because any other SDS of the form \([\text{Par}_{K_{n+1}}, \sigma]\) is topologically conjugate to \([\text{Par}_{K_{n+1}}, \text{id}]\) via the standard \( F_{2}^{n+1} \) permutation action induced by the \( K_{n+1} \) graph automorphism \( \sigma \) (see Ref. 15). We next observe that \( \text{par}_n: F_2^n \to F_2 \) satisfies the following functional relation:

\[
\psi(x_1, \ldots, x_{n-1}, \psi(x_1, \ldots, x_n)) = x_n. \tag{2.6}
\]

As a consequence of this, we derive

\[
x = (x_1, x_2, \ldots, x_n) \mapsto (\text{par}_n(x), x_2, x_3, \ldots, x_n)
\]

\[
\mapsto (\text{par}_n(x), \text{par}_n(\text{par}_n(x), x_2, \ldots, x_n), x_3, \ldots, x_n)
\]

\[
= (\text{par}_n(x), x_1, x_3, \ldots, x_n)
\]

\[
\vdots
\]

\[
\mapsto (\text{par}_n(x), x_1, x_2, \ldots, x_{n-1}),
\]

where \( \mapsto \) denotes the update of state \( x_i \). We therefore obtain the commutative diagram

\[
\begin{array}{ccc}
F_2^n & \xrightarrow{[\text{Par}_{K_{n+1}}, \text{id}]} & F_2^n \\
\downarrow{\text{proj}} & & \downarrow{\text{proj}} \\
\hat{F}_2^n & \xrightarrow{\sigma_{n+1}} & \hat{F}_2^n
\end{array}
\tag{2.7}
\]

where

\[
\text{proj}(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n),
\]

\[
\iota_{\text{par}_n}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \text{par}_n(x_1, \ldots, x_n)),
\]

\[
\sigma_{n+1}(x_1, x_2, \ldots, x_{n+1}) = (x_{n+1}, x_1, \ldots, x_n),
\]
and \( \hat{F}_2^n = \{ x \in F_2^{n+1} \mid x_{n+1} = \text{par}_n(x_1, \ldots, x_n) \} \). Note that \( \text{proj}: \hat{F}_2^n \rightarrow F_2^n \) and \( \text{par}_n: F_2^n \rightarrow \hat{F}_2^n \) are inverse with respect to each other. Similarly we obtain \( [\text{Par}_{K_n}, \text{id}]^{(2)}(x) = (x_n, \text{par}_n(x), x_1, x_2, \ldots, x_{n-2}) \) and in general \( [\text{Par}_{K_n}, \text{id}]^{(k)} = \text{proj} \circ \sigma_{n+1}^k \circ \text{par}_n \), hence the order of an orbit of \( [\text{Par}_{K_n}, \text{id}] \) is a divisor of \( n + 1 \). Furthermore, it is easy to see that the orbit containing the state \((1, 0, 0, \ldots, 0)\) always has length \( n + 1 \). Explicitly we have for \( n = 7 \):

\[
\begin{array}{cccc}
(1000000) & \rightarrow & (1100000) & \rightarrow (0110000) \rightarrow (0011000) \\
(0000001) & \rightarrow & (0000011) & \rightarrow (0000110) \rightarrow (0001100)
\end{array}
\]

Thus we can deduce from Theorem 1 that \( [\text{Par}_{Q_2^3}, \eta_\phi(\pi)] \) has a periodic orbit of length \( n + 2 \) for any \( \pi \in S_{n+1} \), and the proof of the proposition is complete. \( \square \)

The above proposition raises the following question: Given a graph \( Y \), what are the covering maps \( \psi: Y \rightarrow Z \), and what is the structure of the corresponding graphs \( Z \)? This question is investigated for circle graph and \( n \)-cubes in the next section.

3. Covering Maps

In view of Theorem 1, it is of interest to study covering maps \( \psi: Y \rightarrow Z \). In the following we will analyze this question for circle graphs, i.e. \( Y = \text{Circ}_n \) and \( n \)-cubes \( Y = Q_n^3 \). We close the section with examples of covering maps from non-regular graphs and an informal discussion of techniques for extending a given covering map from a graph \( Y \) to a whole class of “extended” graphs. We remark that our study of covering maps departs somewhat from the usual study of covering maps found in the literature. Where a common covering for two given graphs \( Y \) and \( Z \) is usually sought, we study covering maps defined over a fixed, given graph \( Y \).

We would also like to point out that the example of \( \phi: Q_3^2 \rightarrow K_4 \) is well-known. It has been studied in Ref. 5 through the framework of chain-maps. In the terminology of Ref. 5 we note that \( Q_3^2 \cong \hat{K}_4(\mathbb{F}_2, \phi) \) and that the graph in the third row in Fig. 7 is isomorphic to the covering graph \( K_8(\mathbb{F}_2, \phi) \) through \( \mathbb{F}_2 \)-chains assigning 1 to each arc of \( K_4 \) and \( K_8 \), respectively.

The following result will be used in the next sections:

**Lemma 1.** Let \( Y \) and \( Z \) be non-empty, connected graphs, and let \( \phi: Y \rightarrow Z \) be a covering map. Then \( \phi \) is surjective and

\[
\forall x, y \in v[Z]: \ |\phi^{-1}(x)| = |\phi^{-1}(y)|.
\]

In particular, \( |Y| \equiv 0 \mod |Z| \).

**Proof.** Assume there exists \( x, y \in v[Z] \) such that \( |\phi^{-1}(x)| > |\phi^{-1}(y)| \). Since \( Z \) is connected we may without loss of generality assume that \( \{x, y\} \in e[Z] \). By local
bijectivity we deduce that for each $\xi_i \in \phi^{-1}(x)$ there is exactly one edge in $Y$ of the form $\{\xi_i, \eta_i\}$ where $\eta_i \in \phi^{-1}(y)$, and the lemma follows.

Thus, when the conditions of the lemma are satisfied the constant $\kappa_\phi = |\phi^{-1}(x)|$, $x \in v[Z]$ is well-defined, i.e. it does not depend on $x \in v[Z]$. It is the fiber depth of covering map $\phi$. The lemma can be formulated for graphs that are not connected. In this case it states that the fiber depth is constant on components of $Z$. However, this more general formulation is not needed in here.

### 3.1. Covering maps over $\text{Circ}_n$

We label the vertices of $\text{Circ}_n$ by $0, 1, \ldots, n-1$, e.g. the elements of $\mathbb{Z}_n$. Elements of $\text{Aut}(\text{Circ}_n)$ are of the form $\tau^m \sigma^k$ where $\sigma = (0, 1, \ldots, n-1)$ and $\tau = \prod_{i=0}^{n-1} (i, n-i)$. The covering maps from $\text{Circ}_n$ are characterized by the following:

**Proposition 2.** Let $\gamma: \text{Circ}_n \to Z$ be a locally surjective graph morphism. Then $Z$ is either a line graph or circle graph. For any covering image $Z$ of $\text{Circ}_n$ there is a subgroup $H < \text{Aut}(\text{Circ}_n)$ and a covering map $\gamma$ given by

$$\text{Circ}_n \xrightarrow{\gamma} H \backslash \text{Circ}_n \cong Z. \quad (3.1)$$

In particular, there are no non-trivial covering maps for $n < 6$.

**Proof.** Assume $\gamma: Y \to Z$ is a covering map and that $Y$ is connected. It is clear that paths in $Y$ map into (multi-)paths in $Z$, so $Y$ being connected implies that $Z$ is connected. In particular, any covering image of $\text{Circ}_n$ is connected. By local bijectivity of $\gamma: \text{Circ}_n \to Z$ any vertex $i$ in $Z$ has degree two. Thus any covering image of $\text{Circ}_n$ is a connected regular graph of degree two from which we conclude that it is (isomorphic to) a circle graph.

Let $Z$ be a covering image of $\text{Circ}_n$. We know that $Z$ is a circle graph and that the size $m$ of $Z$ divides $n$, $m \geq 3$ by Lemma 1. The subgroup $H = \langle \sigma^m \rangle$ satisfies $Z \cong H \backslash \text{Circ}_n$ and gives us the desired covering map by $\gamma = \pi_H$,

$$\pi_{\langle \sigma^m \rangle}: \text{Circ}_n \to \langle \sigma^m \rangle \backslash \text{Circ}_n \cong Z.$$

Next assume that $\gamma$ is locally surjective but not locally bijective. Then there is at least one vertex $\gamma(i_0)$ in $Z$ that has degree one. As a consequence, the neighbors of $i_0$ in $Y$, say $i_1$ and $i'_1$, are both mapped to the same vertex $\gamma(i_1)$ in $Z$. The neighbors of $i_1$ and $i'_1$ different from $i_0$, which we denote by $i_2$ and $i'_2$, are also mapped to the same vertex $\gamma(i_2)$ in $Z$, as otherwise the degree of $\gamma(i_1)$ would be 3, which is impossible. If $n$ is even, say $n = 2m$, then this argument terminates with the single vertex $i_m$ whose image has degree one, and we have a line graph. If $n$ is odd, say $n = 2m + 1$, then the argument ends with two adjacent vertices $i_m$ and $i'_m$ being mapped to the same vertex $\gamma(i_m)$ which then would have degree three. Consequently, there exists no locally surjective graph morphism that is not locally...
bijective for \( n \) odd. Let \( Z \) be the image of the surjective graph morphism. Then we have

\[
\pi(\tau) : \text{Circ}_n \to \langle \tau \rangle \backslash \text{Circ}_n \cong Z \cong \text{Line}_{n/2}.
\]

The last statement of the proposition follows from Lemma 1 and the fact that for every covering we have \( d(i,j) \geq 3 \) for any \( i, j \in Y \) with \( i \neq j \) and \( i, j \in \gamma^{-1}(\gamma(i)) \).

**Example 3.** We consider SDS induced by the parity function over \( \text{Circ}_3 \), \( \text{Circ}_6 \) and \( \text{Circ}_{12} \) as shown in Fig. 5. In light of Proposition 2 above we have covering maps \( \psi : \text{Circ}_{12} \to \text{Circ}_3 \), \( \psi_1 : \text{Circ}_{12} \to \text{Circ}_6 \) and \( \psi_2 : \text{Circ}_6 \to \text{Circ}_3 \). Let \( \sigma_{12} = (0, 1, 2, \ldots, 11) \) and \( \sigma_6 = (0, 1, \ldots, 5) \) where we use cycle notation for permutations. The map \( \psi \) is induced by \( \sigma_{12}^3 \) while \( \psi_1 \) is induced by \( \sigma_{12}^6 \) and \( \psi_2 \) is induced by \( \sigma_6^3 \). We have shown in Ref. 15 that SDS induced by parity are invertible, and consequently all states are contained in phase space orbits. Hence we only need to consider orbit sizes here. Starting with the schedule \( \pi = (0, 1, 2) \) (identity, no cycle notation) we lift as follows:

\[
\pi_1 = (0, 1, 2) \rightarrow \pi_2 = (0, 3, 1, 4, 2, 5) \rightarrow (0, 3, 6, 9, 1, 4, 7, 10, 2, 5, 8, 11) \sim_{\text{Circ}_{12}} \pi_3 = (0, 6, 3, 9, 1, 7, 4, 10, 2, 8, 5, 11)
\]

We have an embedding of \( \Gamma[\text{Par}_{\text{Circ}_3}, \pi_1] \) into both \( \Gamma[\text{Par}_{\text{Circ}_6}, \pi_2] \) and \( \Gamma[\text{Par}_{\text{Circ}_{12}}, \pi_3] \) as well as an embedding of \( \Gamma[\text{Par}_{\text{Circ}_6}, \pi_2] \) into \( \Gamma[\text{Par}_{\text{Circ}_{12}}, \pi_3] \). The orbit sizes occurring in each of the three systems is summarized in Table 1.

The following result combined with Proposition 2 yields insight in the structure of parity SDSs over circle graphs. As we mentioned previously, SDSs induced by
parity, or the logical complement of parity, are the only invertible SDSs induced from symmetric functions [3].

**Proposition 3.** The SDS $([\text{Par}_{\text{Circ}}, \pi])$: $F_2^n \rightarrow F_2^n$ is conjugate to a right-shift of length $n - 2$ on $F_2^{2n-2}$. Let $\text{Per}(x)$ denote the orbit length of the orbit containing the periodic point $x$. We have in particular,

$$\text{Per}(x) = \begin{cases} 0 \mod n - 1, & n \equiv 0 \mod 2 \\ 0 \mod 2n - 2, & \text{else} \end{cases}$$

(3.2)

for all $x \in F_2^n$. The same statement holds for the SDS $[\text{Par}_{\text{Circ}}, \pi]^\prime$: $F_2^n \rightarrow F_2^n$.

**Proof.** Define the embedding $\iota: F_2^n \rightarrow F_2^{2n-2}$ by

$$\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_n + x_1 + x_2, x_n + x_1 + x_3, \ldots, x_n + x_1 + x_{n-1}),$$

and set $\widehat{F}_2^{2n-2} = \iota(F_2^n)$. A direct calculation shows that the diagram

$$\begin{array}{c}
\begin{array}{c}
{F_2^n}
\end{array}
\downarrow
\begin{array}{c}
\iota
\end{array}

\begin{array}{c}
[\text{Par}_{\text{Circ}}, \text{id}]
\end{array}
\downarrow
\begin{array}{c}
\iota
\end{array}

\begin{array}{c}
\widehat{F}_2^{2n-2}
\end{array}
\end{array}
\quad (3.3)
$$

commutes. Here $\sigma_{n-2}: \widehat{F}_2^{2n-2} \rightarrow \widehat{F}_2^{2n-2}$ is defined by

$$\sigma_{n-2}(x_1, \ldots, x_{2n-2}) = (x_{n+1}, \ldots, x_{2n-1}, x_1, \ldots, x_n).$$

It is well-defined. Note that $\iota: F_2^n \rightarrow \widehat{F}_2^{2n-2}$ is a bijection. Thus the map $\sigma$ and $[\text{Par}_{\text{Circ}}, \text{id}]$ are topologically conjugate (discrete topology) under $\iota$.

Explicitly, we have

$$[\text{Par}_{\text{Circ}}, \text{id}](x_1, x_2, \ldots, x_n)
= (x_n + x_1 + x_2, x_n + x_1 + x_3, \ldots, x_n + x_1 + x_{n-1}, x_1, x_2),$$

and then

$$\iota(x_n + x_1 + x_2, x_n + x_1 + x_3, \ldots, x_n + x_1 + x_{n-1}, x_1, x_2)
= (x_n + x_1 + x_2, x_n + x_1 + x_3, \ldots, x_n + x_1 + x_{n-1}, x_1, x_2, x_3, \ldots, x_n).$$

On the other hand, this also equals $(\sigma_{n-2} \circ \iota)(x_1, \ldots, x_n)$, verifying the commutative diagram.
From the conjugation relation it is clear that the size of a periodic orbit under $[\text{Par}_{\text{Circ}}, \text{id}]$ must be a divisor of $(2n - 2) / \gcd(n - 2, 2n - 2)$. The statement of the proposition follows from the fact that

$$\gcd(n - 2, 2n - 2) = \begin{cases} 1, & n \equiv 0 \mod 2 \\ 2, & \text{else.} \end{cases}$$

The proof for $[\text{Par}_{\text{Circ}}, \text{id}] : F_2^n \to F_2^n$ is analogous.

### 3.2. Covering maps over $Q_2^n$

The covering map $\phi : Q_2^n \to K_4$ given in the introduction is a special instance of a whole class of covering maps over generalized $n$-cubes. The key approach is to consider $Q_2^n$ as a Cayley graph and $K_4$ as an orbit space with respect to a regularly acting subgroup of $Q_2^n$-automorphisms.

The following result is analogous to Proposition 2 and studies the relation between automorphisms and covering maps for binary $n$-cubes.

**Theorem 2.** For any subgroup $H' < F_2^n$ with $[F_2^n : H'] \geq n + 1$ there exists an isomorphic subgroup $H \cong H'$ with the property

$$H(x) \cap H(y) = \emptyset \quad \text{for } x \neq y; \quad x, y \in \{0, e_1, \ldots, e_n\}.$$  \hspace{1cm} (3.4)

For any subgroup $H < F_2^n$ with property (3.4) the graph $H \setminus Q_2^n$ is connected, undirected and loop-free, and the natural projection

$$\pi_H : Q_2^n \to H \setminus Q_2^n, \quad v \mapsto H(i)$$

is a covering-map.

This result is a special case of Proposition 3 in Ref. 18. The result given there proves the result for generalized $n$-cubes. We refer to Ref. 18 for the proof.

**Example 4 (Covering Maps Over $Q_2^4$ and $Q_2^7$).** We have constructed all covering images of the form $H \setminus Q_2^n$ for $n = 4$ and $n = 7$. There are two non-isomorphic covering images when $n = 4$ and five non-isomorphic covering images of size 16 when $n = 7$. For $n = 4$ explicit computations show that the two covering images of the form $H \setminus Q_2^4$ are the only covering images from $Q_2^4$. In other words, all covering maps from $Q_2^4$ can be represented through an appropriate subgroup of $F_2^4$.

The covering images for $n = 4$ and $n = 7$ (size 16) are shown in Figs. 6 and 7, respectively.

The following corollary is a consequence of the proof of Proposition 3 in Ref. 18.

**Corollary 1.** Let $n$ be a natural number. Then we have $2^n \equiv 0 \mod n + 1$ if and only if there exists a subgroup $G < F_2^n$ with the property $F_2^n = G(0) \cup \bigcup_{i=1}^n G(e_i)$.

We are now in the position to prove the existence of covering maps $\phi : Q_2^n \to K_{n+1}$ whenever we have $2^n \equiv 0 \mod n + 1$. This result was used in the proof of Proposition 1.
Fig. 6. All (non-isomorphic) covering image graphs of $Q_2^4$.

Fig. 7. All non-isomorphic graphs of the form $H\setminus Q_2^7$ on 16 vertices. All graphs have $Q_2^4$ as a subgraph. The stippled edges show how the inner and outer cubes are connected. The other vertices of the outer cube are connected in the same way, but the edges are omitted. Note that there are 4 lines in the middle figure on the top (marked with ⋆). The two lines connecting the outer cube to the extreme points of the inner cube coincide.

**Corollary 2.** Let $n$ be a natural number. Then there exists a covering map

$$\varphi: Q_2^n \rightarrow K_{n+1}$$

if and only if $2^n \equiv 0 \mod n + 1$.

**Proof.** Let $\phi: Q_2^n \rightarrow K_{n+1}$. From Lemma 1 we immediately conclude $2^n \equiv 0 \mod n + 1$. Suppose next that $2^n \equiv 0 \mod n + 1$. Lemma 1 guarantees
the existence of a subgroup $G < \mathbb{F}_2^n$ such that $\mathbb{F}_2^n = G(0) \cup \bigcup_{i=1}^n G(e_i)$. Property (3.4) is clearly fullfilled for $G$ and from Theorem 2 we obtain a covering map $\phi: \mathbb{Q}_2^n \to G\setminus Q_2^n$ given by
\[
\phi^{-1}(i) = G(e_i), \quad 1 \leq i \leq n, \quad \phi^{-1}(n+1) = 0.
\] (3.6)

It remains to establish that $K_{n+1} \cong G\setminus Q_2^n$. Obviously, $G\setminus Q_2^n$ contains $n + 1$ vertices. Let $i \neq j$ be two $G\setminus Q_2^n$ vertices and $x \in G(e_i)$. We observe that each of the sets $G(0)$, $G(e_s)$ for $s = 1, \ldots, n$ is dense in $Q_2^n$, from which we can conclude that there exists some $y \in G(e_j)$ that is adjacent to $x$. Accordingly, any two distinct $G\setminus Q_2^n$ vertices are adjacent in $G\setminus Q_2^n$ from which we conclude that $G\setminus Q_2^n \cong K_{n+1}$. $\square$

3.3. Covering maps over non-regular graphs

The previous two subsections have dealt with covering maps from regular graphs. Here we give an example of covering maps from non-regular graphs. In fact, the principle of edge splitting used here is quite general and can be extended as in Ref. 14.

Example 5. The extended 3-cube $EQ_3^2$ is obtained from the binary 3-cube by inserting a vertex on each edge, see Fig. 8 where the extended cube is shown in both diagrams on the left. The covering map $\phi$ as given in Fig. 8 maps $EQ_3^2$ into the extended complete graph on four vertices, $EK_4$. There is also covering map $\phi_1$ into another ten vertex graph that is non-isomorphic to $EK_4$. This other ten vertex graph has a covering map $\phi_2$, with image graph say $Z$, as shown in the lower diagram of Fig. 8 but there are no covering maps from $EK_4$. Using the schedule $\pi = ((000), a, b, c, (100))$ for the SDS over $Z$ induced by parity we obtain eight fixed points, four 2-orbits and four 4-orbits, all of which embed into the phase space of the SDS over $EQ_3^2$ with schedule
\[
((000), (110), (101), (011), (100), (010), (001), (111), \newline
a_1, a_2, a'_1, a'_2, b_1, b_2, b'_1, b'_2, c_1, c_2, c'_1, c'_2, d_1, d_2, d'_1, d'_2).
\]

We refer to the upcoming paper, Ref. 14, for more on this.

4. Summary

In the paper we have studied specific cases of how to construct smaller SDSs for a given SDS, and how to use the derived SDSs to obtain insight into the phase space structure of our original SDS. The approach is comparable to the application of the Hartman/Grobman theorem of classical dynamical system theory in the following sense: Whenever we have a covering map between graphs we can construct
smaller/simpler SDSs that give us partial information about our system. We also gave existence and structural results for covering maps over, e.g. circle graphs.

Future directions for this research include extending covering maps along the lines alluded to in the last section as well as studying other graph classes than circles and hypercubes. We refer to Ref. 14 where some of this work is already in progress.
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